

VECTOR FIELDS ON QUANTUM GROUPS

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Abstract

We construct the space of vector fields on a generic quantum group. Its elements are products of elements of the quantum group itself with left invariant vector fields. We study the duality between vector fields and 1-forms and generalize the construction to tensor fields. A Lie derivative along any (also non left invariant) vector field is proposed and a puzzling ambiguity in its definition discussed. These results hold for a generic Hopf algebra.

1 Introduction

Following the program of generalizing the differential geometry structures to the non-commutative case, we construct on a Hopf algebra the analog of the space of vector fields.

Indeed in the literature the quantum Lie algebra of left invariant vector fields as well as the space of 1-forms has been extensively analyzed [2, 3, 13, 4, 16, 8, 7], while the notion of generic vector field on a Hopf algebra and the duality relation with the space of 1-forms deserves more study [9, 10].

We will see how left invariant vector fields generate the whole space of vector fields. This space can be also characterized as the bicovariant bimodule (vector bundle) dual to that of 1-forms.

Throughout this paper we will deal with a Hopf algebra A [11] over \mathbf{C} with coproduct $\Delta : A \rightarrow A \otimes A$, counit $\varepsilon : A \rightarrow \mathbf{C}$ and invertible antipode $S : A \rightarrow A$. Particular cases of Hopf algebras are quantum groups, which for us will be Hopf algebras with one (or more) continuous parameter q ; when $q=1$ the product “ \cdot ” in A becomes commutative and we obtain the algebra of functions on a group. When we will speak of commutative case we will refer to the Hopf algebra $C^\infty(G)$ of smooth functions on a (compact) Lie group G .

In Section 2 we briefly recall how to associate a Connes-type differential calculus to a given Hopf algebra and we emphasize the role played by the tangent vectors. This construction will be effected along the lines of Woronowicz’ work [2]. Indeed the results in [2] apply also to a general Hopf algebra with invertible antipode (not necessary a compact matrix pseudogroup). This can be shown by checking that all the formulae used for the construction — collected in the appendix of [2] — hold also in the case of a Hopf algebra with invertible antipode ¹.

While following the work of [2] in spirit, we however decided on a small but important change of conventions [13] in this presentation. It is based on the following observation: while not a priori obvious, it is indeed possible to write all defining relations of the differential calculus as *deformed commutation relations*. Now, given that this is possible, we would like to have operators acting from the left and to the right as one is used to. This made the change in conventions necessary. Conversion to the old conventions (denoted by a “W” subscript) is possible in two ways: either one substitutes

$$\chi_i = -S^{-1}({}_W\chi_i), \quad f_j^i = S^{-1}({}_Wf_j^i), \quad etc.$$

into our equations as will be mentioned in the text, or one simply reads all equations “from the left to the right”. Notice that the ad-invariant right ideal R , the differential and the forms are the same. For quick reference we would like to point out the first version [12] of this article, where the old convention was used.

¹Formula (A.22) in [2] is the most difficult to prove and necessitates the further axiom of the invertibility of the antipode S ; also the invertibility of the map s in (A.18) relies on the existence of S^{-1} . All the other formulae are direct consequences of the Hopf algebra axioms.

In Section 3 we construct the space of vector fields, while in Section 4 we study the action of the Hopf algebra on the vector fields; i.e. we will study the push-forward of vector fields on Hopf algebras. Then we deal with (covariant and contravariant) tensor fields and wedge products.

In the last three sections we introduce and discuss Lie derivatives and a contraction operator on differential forms along generic vector fields. These two operators are basic tools for the formulation of deformed gravity theories [5], where the relevant Lie algebra is the q -Poincaré Lie algebra.

2 Differential Geometry on Hopf Algebras

In the commutative case, given the differential calculus on a (compact) Lie group G , we can consider the subspace in the space of all smooth functions $f : G \rightarrow \mathcal{C}$ defined by:

$$R \equiv \{h \in C^\infty(G) \mid h(1_G) = 0 \text{ and } dh(1_G) = 0\} , \quad (2.1)$$

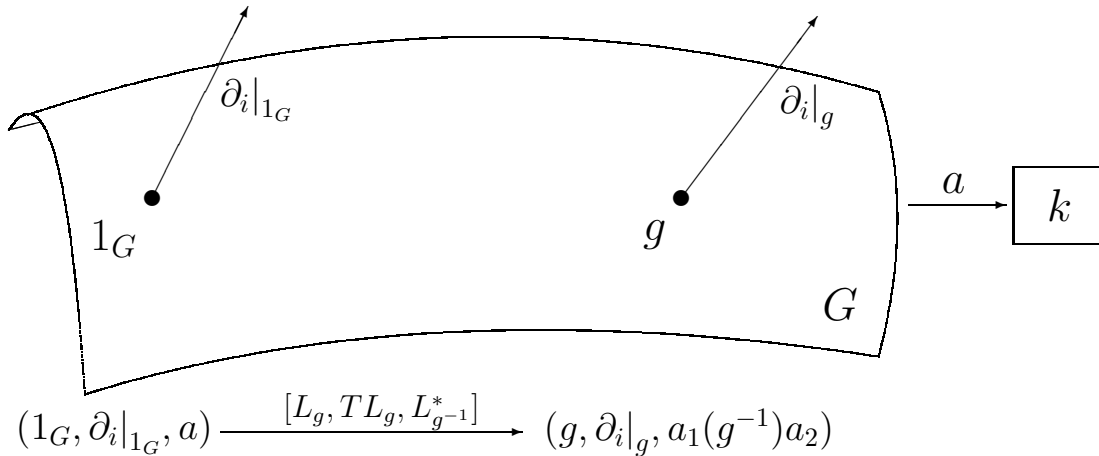
where 1_G is the unit of the group.

R is a particular ideal of the Hopf algebra $C^\infty(G)$ namely $(\ker \epsilon)^2$. All the information about the differential calculus on G is contained in R .

Indeed the space of tangent vectors at the origin of the group is given by all the linear functionals which annihilate R and any constant function. Locally we write a basis as $\{\partial_i|_{1_G}\}$. Once we have this basis, using the tangent map (namely TL_g) induced by the left multiplication of the group on itself: $L_g g' = gg'$, $\forall g, g' \in G$ we can construct a basis of left invariant vector fields $\{t_i\}$. The action of these vector fields on a generic function a on the group manifold is (here and in the following: $\Delta(a) \equiv a_1 \otimes a_2$ is the coproduct in Sweedler's notation [11])

$$t_i(a) = a_1(\partial_i a_2|_{1_G}) \equiv \partial_i|_{1_G} * a$$

in compliance with the following picture:



Note that $L_g^*(a)(h) := a(gh) = a_1(g)a_2(h)$.

A generic 1-form can be written $\rho = \omega^i f_i$ [$f_i \in C^\infty(G)$] where $\{\omega^i\}$ is the dual basis of $\{t_i\}$. Finally, the differential on functions is

$$d = \omega^i t_i \quad \text{that is} \quad df = \omega^i t_i(f) . \quad (2.2)$$

In [2] the quantum analog of $R \subset A$ is studied. Similarly to the classical case R is a right ideal of A , formally $RA \subset R$, it is Ad-invariant, *i.e.* $\forall r \in R : r_2 \otimes S(r_1)r_3 \in R \otimes A$, and its elements have vanishing counit. Given a Hopf algebra A , it turns out that we can always find an R and construct a differential calculus (in general not unique).

The space of tangent vectors on A is then defined as:²

$$T \equiv \{\chi : A \rightarrow \mathbf{C} \mid \chi \text{ linear functionals, } \chi(I) = 0 \text{ and } \chi(SR) = 0\} ,$$

where I is the unit of A (in the commutative case it is the constant function $I(g) = 1 \forall g \in G$). In the sequel $\{I\}$ is the linear span of I . Let $\{\chi_i\}$ $i = 1, \dots, n$ be a basis of T . Consider the linear space X such that

$$A = X \oplus R \oplus \{I\} \quad (2.3)$$

X is maximal in the (ordered) set of all linear subspaces of A disjoint from $R \oplus \{I\}$. From (2.3) it follows that the dual vector space X^* is isomorphic to T and therefore there are n elements $x^i \in X \subset \ker \varepsilon$ uniquely defined by the duality

$$\chi_i(x^j) = \delta_i^j . \quad (2.4)$$

Note that $\varepsilon(x^i) = 0$ since $X \subset \ker \varepsilon$ because $A = \ker \varepsilon \oplus \{I\}$. We can then define the n^2 linear functionals³ $f_j^i : A \longrightarrow \mathbf{C}$

$$\forall a \in A \quad f_j^i(a) \equiv \chi_j(ax^i) , \quad (2.5)$$

and (2.4)] and we have (from $AS(R) = S(R)$)

$$\chi_i(ab) = \chi_i(a)\varepsilon(b) + f_j^i(a)\chi_j(b) . \quad (2.6)$$

This is the deformed Leibniz rule for the operators χ_i . In the $q = 1$ case, when R becomes the set defined in (2.1), we have $\chi_i = \partial_i|_{1_G}$, $f_j^i = \delta_j^i \varepsilon$ and we write (2.6) as $\partial_i(fh)|_{1_G} = (\partial_i f|_{1_G})h(1_G) + f(1_G)(\partial_i h|_{1_G})$.

For consistency with (2.6) the f_j^i must satisfy the conditions:

$$f_j^i(ab) = f_j^k(a)f_k^i(b) \quad (2.7)$$

$$f_j^i(I) = \delta_j^i . \quad (2.8)$$

²There are two tangent spaces:

The one we have chosen and the one corresponding to $\chi(R) = 0$ [2].

³Relation to the conventions of [2] (denoted by a “W” subscript):
 $\chi_i = -S^{-1}_w \chi_i$, $f_j^i = S^{-1}_w f_j^i$, $R = {}_w R$, $x^i = -S_w x^i$.

The space of left invariant vector fields ${}_{inv}\Xi$ is easily constructed from T . Using the coproduct Δ we define — as in the commutative case — $\chi * a = (id \otimes \chi)\Delta(a) \equiv a_1\chi(a_2)$ and

$${}_{inv}\Xi \equiv \{t \mid t = \chi * \text{ where } \chi \in T\} \quad (2.9)$$

There is a one to one correspondence $\chi_i \leftrightarrow t_i = \chi_i *$. In order to obtain χ_i from $\chi_i *$ we simply apply ε :

$$(\varepsilon \circ t_i)(a) = \varepsilon(a_1)\chi_i(a_2) = \chi_i(a), \quad (2.10)$$

where we have used the Hopf algebra axioms.

${}_{inv}\Xi$ is the vector subspace of all linear maps from A to A that is isomorphic to T .

We have chosen this perspective to introduce the space of left invariant vector fields in order to point out that (also in the case of a general Hopf algebra) it has an existence on its own, independent of the space of 1-forms.

The space of 1-forms Γ is formed by all the elements ρ that are written as formal products and sums of the type

$$\rho = \omega^i a_i . \quad (2.11)$$

Here $a_i \in A$ and ω^i $i = 1, \dots, n$ is the basis dual to $\{t_i\}$. We express this duality with a bracket:

$$\langle \chi_j, \omega^i \rangle = \delta_j^i . \quad (2.12)$$

Relation (2.11) tells us that the space of 1-forms is freely generated by the elements ω^i . By definition any ρ is decomposable in a unique way as $\rho = \omega^i a_i$ and Γ is a right A -module with the trivial product $(\omega^i a_i)b \equiv \omega^i(a_i b)$. Γ is also a left A -module with the following left product:

$$\forall b \in A \quad b\omega^i = \omega^j(f_j^i * b) \equiv \omega^j(id \otimes f_j^i)\Delta(b) . \quad (2.13)$$

From this relation it follows that:

$$\forall a \in A \quad \omega^i a = [(f_j^i \circ S) * a]\omega^j = a_1 f_j^i(Sa_2)\omega^j \quad (2.14)$$

and that any ρ can be written in a unique way in the form

$$\rho = b_i \omega^i \quad (2.15)$$

with $b_i \in A$.

Finally, the differential operator $d : A \longrightarrow \Gamma$ can be defined through the relation:

$$\forall a \in A \quad da = \omega^i(\chi_i * a) . \quad (2.16)$$

Notice that this can be rewritten:

$$da = \omega^i(\chi_i * a) = S f_i^j * (\chi_j * a)\omega^i = (-S\chi_i * a)\omega^i \quad (2.17)$$

where we have used (2.14) and (2.6), i.e. ${}_w\chi_i = -S\chi_i$. As a consequence, the differential calculus obtained has the following properties [2]:

i) The differential operator satisfies the Leibniz rule

$$d(ab) = (da)b + a(db) \quad \forall a, b \in A. \quad (2.18)$$

Moreover any $\rho \in \Gamma$ can be expressed as

$$\rho = da_\alpha b_\alpha \quad (2.19)$$

for some a_α, b_α belonging to A . (Use $\omega^i = dx_2^i S^{-1} x_1^i$ below).

Remark. Once we know the operator d , the space of tangent vectors on A , like in the commutative case, can be defined as:

$$T = \{\chi \mid \chi(a) = 0 \text{ if and only if } Pda = 0\} \quad (2.20)$$

where $Pda \equiv da_2 S^{-1} a_1$ with $\Delta(a) = a_1 \otimes a_2$. The linear map P is a projection operator; to a given form $\rho = \omega^i a_i$ it associates the form $P(\rho) = \varepsilon(a_i) \omega^i$ and in particular $P(dx^i) = \omega^i$. In the commutative case $\varepsilon(a_i)$ is the value that $a_i \in A = C^\infty(G)$ takes in the origin 1_G of the Lie group G . $P(\rho)$ is then the left invariant 1-form whose value in the origin 1_G of the Lie group equals the value of the 1-form ρ in 1_G .⁴

ii) The differential calculus is called bicovariant because using d and the coproduct Δ we can define two linear compatible maps Δ_Γ and ${}_\Gamma\Delta$

$$\Delta_\Gamma(da) = (id \otimes d)\Delta(a), \quad \Delta_\Gamma : \Gamma \rightarrow A \otimes \Gamma \quad (\text{left covariance}) \quad (2.21)$$

$${}_\Gamma\Delta(da) = (d \otimes id)\Delta(a), \quad {}_\Gamma\Delta : \Gamma \rightarrow \Gamma \otimes A \quad (\text{right covariance}) \quad (2.22)$$

$$\Delta_\Gamma(a\rho b) = \Delta(a)\Delta_\Gamma(\rho)\Delta(b), \quad {}_\Gamma\Delta(a\rho b) = \Delta(a){}_\Gamma\Delta(\rho)\Delta(b) \quad (2.23)$$

which represent the left and right action of the Hopf algebra A on Γ . In the commutative case they express the pull-back on 1-forms induced by the left or right multiplication of the group on itself [7]. Δ_Γ and ${}_\Gamma\Delta$ are compatible in the sense that $(id \otimes {}_\Gamma\Delta)\Delta_\Gamma = (\Delta_\Gamma \otimes id){}_\Gamma\Delta$. In the commutative case this formula tells us that the left and right actions of the group on Γ commute: $R_g^* L_{g'}^* = L_{g'}^* R_g^* \quad \forall g, g' \in G$. From the definitions (2.21) and (2.22) one deduces the following properties [2]:

$$(\varepsilon \otimes id)\Delta_\Gamma(\rho) = \rho, \quad (id \otimes \varepsilon){}_\Gamma\Delta(\rho) = \rho \quad (2.24)$$

$$(\Delta \otimes id)\Delta_\Gamma = (id \otimes \Delta_\Gamma){}_\Gamma\Delta, \quad (id \otimes \Delta){}_\Gamma\Delta = ({}_ \Gamma\Delta \otimes id){}_\Gamma\Delta. \quad (2.25)$$

An element ω of Γ is said to be *left invariant* if

$$\Delta_\Gamma(\omega) = I \otimes \omega \quad (2.26)$$

⁴In [2]: ${}_wPda = S(a_1)da_2$, such that $\omega^i = Pdx^i = {}_wPd_w x^i = {}_w\omega^i$.

and *right invariant* if

$${}_r\Delta(\omega) = \omega \otimes I \quad (2.27)$$

We have seen that any ρ is of the form $\rho = \omega^i a_i$. We have that the ω^i are left invariant and form a basis of ${}_{inv}\Gamma$, the linear subspace of all left invariant elements of Γ . Relation (2.12) tells us that ${}_{inv}\Gamma$ and ${}_{inv}\Xi$ are dual vector spaces.

iii) There exists an *adjoint representation* M_j^i of the Hopf algebra, defined by the right action on the ω^i :

$${}_r\Delta(\omega^i) = \omega^j \otimes M_j^i ; \quad M_j^i \in A . \quad (2.28)$$

The co-structures on the M_j^i can be deduced [2]:

$$\Delta(M_j^i) = M_j^l \otimes M_l^i \quad (2.29)$$

$$\varepsilon(M_j^i) = \delta_j^i \quad (2.30)$$

$$S(M_i^j) = (M^{-1})_i^j \quad (2.31)$$

The elements M_j^i can be used to build a right invariant basis of Γ . Indeed the η^i defined by

$$\eta^i \equiv \omega^j S(M_j^i) \quad (2.32)$$

are a basis of Γ (every element of Γ can be uniquely written as $\rho = \eta^i b_i$) and their right invariance can be checked directly .

Moreover, from (2.31), using (2.32) and (2.14) one can prove the relation

$$(a * f_i^j) M_j^k = M_i^j (f_j^k * a) \quad (2.33)$$

with $a * f_i^j \equiv (f_i^j \otimes id)\Delta(a)$ being the action of the right-invariant operator f_i^j on the function a .

3 Construction of the space of Vector Fields.

In this section we study the space Ξ of vector fields over Hopf algebras defining a left product between elements of A and of ${}_{inv}\Xi$.

A generic vector field can be written in the form $f^i \cdot t_i$ where $\{t_i\}$ $i = 1, \dots, n$ is a basis of left invariant vector fields and f^i are n smooth functions on the group manifold. In the commutative case $f^i \cdot t_i = t_{i\Box} f^i$ i.e. left and right products (that we have denoted with \Box) are the same, indeed $(t_{i\Box} f^i)(h) \equiv t_i(h) f^i = f^i t_i(h) = (f^i \cdot t_i)(h)$.

Let $t_i = \chi_i^*$ be a basis in ${}_{inv}\Xi$ and let a^i , $i = 1, \dots, n$ be generic elements of A :

Definition

$$\Xi \equiv \{V \mid V : A \longrightarrow A ; V = a^i \cdot t_i\} , \quad (3.1)$$

where the definition of the left product \cdot is given below:

Definition

$$\forall a, b \in A, \forall t \in_{inv} \Xi \quad (a \cdot t)b \equiv at(b) = a(\chi * b) . \quad (3.2)$$

The product \cdot has a natural generalization to the whole Ξ :

$$\begin{aligned} \cdot : \quad A \times \Xi &\longrightarrow \Xi \\ (a, V) &\longmapsto a \cdot V \end{aligned} \quad \text{where} \quad \forall b \in A \quad (a \cdot V)(b) \equiv aV(b) . \quad (3.3)$$

It is easy to prove that (Ξ, \cdot) is a left A -module:

$$(a + b) \cdot V = a \cdot V + b \cdot V ; \quad (ab) \cdot V = a \cdot (b \cdot V) ; \quad (\lambda a) \cdot V = \lambda a \cdot V \quad (3.4)$$

with $\lambda \in \mathcal{C}$.

Note: The left product was chosen in the definition of Ξ so that the symbol \cdot can henceforth be omitted in all formulae. For the right product \square — that we will introduce later — we *do* have to distinguish for instance the elements $V_{\square}(ab) \in \Xi$ and $V(ab) \in A$ because the vector fields act to the right (by convention).

Ξ is the analog of the space of derivations on the ring $C^\infty(G)$ of the smooth functions on the group G . Indeed we have:

$$V(a + b) = V(a) + V(b) , \quad V(\lambda a) = \lambda V(a) \quad \text{Linearity} \quad (3.5)$$

$$V(ab) \equiv (c^i t_i)(ab) = V(a)b + c^i (f_i^j * a) t_j(b) \quad \text{Leibniz rule} \quad (3.6)$$

in the classical case $c^i (f_i^j * a) t_j(b) = aV(b)$ (recall $f_i^j = \delta_i^j \varepsilon$; $\varepsilon * b = b$).

This last equation can be written as a commutation relation

$$Va = V(a) + c^i (f_i^j * a) t_j \quad (3.7)$$

and in a basis-independent version that uses the right product \square (4.12)⁵

$$Va = V(a) + V_{\square} a. \quad (3.8)$$

We have seen the duality between $_{inv}\Gamma$ and $_{inv}\Xi$. We now extend it to Γ and Ξ , where Γ is seen as a right A -module (not necessarily a bimodule) and Ξ is our left A -module.

Theorem 1. There exists a unique map

$$\langle \ , \ \rangle : \quad \Xi \times \Gamma \longrightarrow A$$

⁵There is another basis-independent version of this equation based on a left U -coaction ${}_U\Delta(V) \equiv V^{1'} \otimes V^2$ (that generalizes the coproduct of $_{inv}\Xi$): $Va = V^{1'}(a)V^2$.

such that:

1) $\forall V \in \Xi$; the application

$$\langle V, \cdot \rangle : \Gamma \longrightarrow A$$

is a right A -module morphism, i.e. is linear and $\langle V, \rho a \rangle = \langle V, \rho \rangle a$.

2) $\forall \rho \in \Gamma$; the application

$$\langle \cdot, \rho \rangle : \Xi \longrightarrow A$$

is a left A -module morphism, i.e. is linear and $\langle bV, \rho \rangle = b\langle V, \rho \rangle$.

3) Given $\rho \in \Gamma$

$$\langle \cdot, \rho \rangle = 0 \Rightarrow \rho = 0, \quad (3.9)$$

where $\langle \cdot, \rho \rangle = 0$ means $\langle V, \rho \rangle = 0 \quad \forall V \in \Xi$.

4) Given $V \in \Xi$

$$\langle V, \cdot \rangle = 0 \Rightarrow V = 0, \quad (3.10)$$

where $\langle V, \cdot \rangle = 0$ means $\langle V, \rho \rangle = 0 \quad \forall \rho \in \Gamma$.

5) On ${}_{inv}\Xi \times {}_{inv}\Gamma$ the bracket $\langle \cdot, \cdot \rangle$ acts as the one introduced in the previous section.

Remark. Properties 3) and 4) state that Γ and Ξ are dual A -moduli, in the sense that they are dual with respect to A .

Proof

Properties 1), 2) and 5) uniquely characterize this map. To prove the existence of such a map we show that the following bracket

Definition

$$\langle V, \rho \rangle \equiv V(a_\alpha)b_\alpha, \quad (3.11)$$

where a_α, b_α are elements of A such that $\rho = da_\alpha b_\alpha$, satisfies 1), 2) and 5).

We first verify that the above definition is well given, that is:

$$\text{Let } \rho = da_\alpha b_\alpha = da'_\beta b'_\beta \quad \text{then} \quad V(a_\alpha)b_\alpha = V(a'_\beta)b'_\beta.$$

Indeed, since

$$da_\alpha b_\alpha = da'_\beta b'_\beta \Leftrightarrow \omega^i t_i(a_\alpha)b_\alpha = \omega^i t_i(a'_\beta)b'_\beta \Leftrightarrow t_i(a_\alpha)b_\alpha = t_i(a'_\beta)b'_\beta$$

[we used the uniqueness of the decomposition (2.11)], the definition is consistent because

$$V(a_\alpha)b_\alpha = V(a'_\beta)b'_\beta \Leftrightarrow c^i t_i(a_\alpha)b_\alpha = c^i t_i(a'_\beta)b'_\beta$$

where $V = c^i t_i$.

Property 1) is trivial since $\rho a = (da_\alpha b_\alpha)a = da_\alpha(b_\alpha a)$.

Property 2) holds since

$$\langle cV, \rho \rangle = (cV)(a_\alpha)b_\alpha = c(V(a_\alpha)b_\alpha) = c\langle V, \rho \rangle .$$

Property 5). Let $\{\omega^i\}$ and $\{t_i\}$ be dual bases in ${}_{inv}\Gamma$ and ${}_{inv}\Xi$. Since $\omega^i \in \Gamma$, $\omega^i = da_\alpha b_\alpha$ for some a_α and b_α in A . We can also write $\omega^i = da_\alpha b_\alpha = \omega^k t_k(a_\alpha)b_\alpha$, so that, due to the uniqueness of the decomposition (2.11), we have

$$t_k(a_\alpha)b_\alpha = \delta_k^i I \quad (I \text{ unit of } A);$$

we then obtain

$$\langle t_j, \omega^i \rangle = t_j(a_\alpha)b_\alpha = \delta_j^i I .$$

Property 3). Let $\rho = \omega^i a_i \in \Gamma$.

If $\langle V, \rho \rangle = 0 \quad \forall V \in \Xi$, in particular $\langle t_j, \rho \rangle = 0 \quad \forall j = 1, \dots, n$; then $\langle t_j, \omega^i \rangle a_i = 0 \Leftrightarrow a_j = 0$, and therefore $\rho = 0$.

Property 4). Let $V = a^i t_i \in \Xi$.

If $\langle V, \rho \rangle = 0 \quad \forall \rho \in \Gamma$, in particular $\langle V, \omega^j \rangle = 0 \quad \forall j = 1, \dots, n$; then $a^i \langle t_i, \omega^j \rangle = 0 \Leftrightarrow a^j = 0$, and therefore $V = 0$.

□□□

By construction every V is of the form

$$V = a^i t_i.$$

We can now show the unicity of such a decomposition.

Theorem 2. Any $V \in \Xi$ can be uniquely written in the form

$$V = a^i t_i$$

Proof

Let $V = a^i t_i = a'^i t_i$ then

$$\forall i = 1, \dots, n \quad a^i = a^j \langle t_j, \omega^i \rangle = \langle V, \omega^i \rangle = a'^j \langle t_j, \omega^i \rangle = a'^i .$$

□□□

Notice that once we know the decomposition of ρ and V in terms of ω^i and t_i , the evaluation of $\langle \quad, \quad \rangle$ is trivial:

$$\langle V, \rho \rangle = \langle b^j t_j, \omega^i a_i \rangle = b^j \langle t_j, \omega^i \rangle a_i = b^i a_i .$$

Vice versa from the previous theorem $V = \langle V, \omega^i \rangle t_i$ and $\rho = \omega_i \langle t_i, \rho \rangle$.

We conclude this section summarizing the three different ways of looking at Ξ :

- (I) Ξ as the set of all deformed derivations over A [see (3.1), (3.5) – (3.8)].
- (II) Ξ as the left A -module freely generated by the *elements* t_i , $i = 1, \dots, n$. The latter is the set of all the *formal* products and sums of the type $a^i t_i$, where a^i are generic elements of A . Indeed, by virtue of Theorem 2, the map that associates to each $V = a^i \cdot t_i$ in Ξ the corresponding element $a^i t_i$ is an isomorphism between left A -moduli.
- (III) Ξ as $\Xi' = \{U : \Gamma \longrightarrow A, U \text{ linear and } U(\rho a) = U(\rho)a \ \forall a \in A\}$, i.e. Ξ as the dual (with respect to A) of the space of 1-forms Γ . The space Ξ' has a trivial left A -module structure: $(aU)(\rho) \equiv aU(\rho)$. Ξ and Ξ' are isomorphic left A -moduli because of property (3.10) which states that to each $\langle V, \cdot \rangle : \Gamma \rightarrow A$ there corresponds one and only one V . [$\langle V, \cdot \rangle = \langle V', \cdot \rangle \Rightarrow V = V'$]. Every $U \in \Xi'$ is of the form $U = \langle V, \cdot \rangle$; more precisely, if a^i is such that $U(\omega^i) = a^i$ then $U = \langle a^i t_i, \cdot \rangle$.

These three ways of looking at Ξ will correspond to different aspects of the Cartan Calculus: the Lie derivatives ℓ_V will generalize (I), inner derivations i_V will correspond to (III), while the transformation properties of ℓ_V and i_V are governed by (II).

4 Bicovariant Bimodule Structure.

In [2] the space of 1-forms is extensively studied. A right and a left product are introduced between elements of Γ and of A , and it is known how to obtain a right product from a left one [e.g. $a\omega^i = \omega^j(f_j^i * a)$, $\omega^i a = (Sf_j^i * a)\omega^j$], i.e. Γ is a bimodule over A .

Since the actions Δ_Γ and ${}_\Gamma\Delta$ are compatible in the sense that:

$$(id \otimes {}_\Gamma\Delta)\Delta_\Gamma = (\Delta_\Gamma \otimes id){}_\Gamma\Delta \quad (4.1)$$

the bimodule Γ is called a *bicovariant bimodule*. In [2] it is shown that relations (2.7), (2.8), (2.14), (2.33), (2.29), (2.30) and (2.28) completely characterize the bicovariant bimodule Γ .

In Section 3 we have studied the left product \cdot and we have seen that Ξ is a left bimodule over A [see (3.4)]. In this section we introduce a right product and a left and right action of the Hopf algebra A on Ξ . The left and right actions Δ_Ξ and $\Xi\Delta$ are the q -analog of the push-forward of tensor fields on a group manifold. Similarly to Γ also Ξ is a bicovariant bimodule.

The construction of the right product on Ξ , of the right action $\Xi\Delta$ and of the left action Δ_Ξ will be effected along the lines of Woronowicz' Theorem 2.5 in [2], whose statement can be explained in the following steps:

Theorem 3. Consider the symbols t_i ($i = 1, \dots, n$) and let Ξ be the left A -module freely generated by them:

$$\Xi \equiv \{a^i t_i \mid a^i \in A\}$$

Consider functionals $O_i^j : A \longrightarrow \mathbf{C}$ satisfying [see (2.7) and (2.8)]

$$O_i^j(ab) = O_i^k(a)O_k^j(b) \quad (4.2)$$

$$O_i^j(I) = \delta_j^i \quad . \quad (4.3)$$

Introduce a right product via the definition [see (2.14)]

Definition

$$(a^i t_i) \circ b \equiv a^i (O_i^j * b) t_j. \quad (4.4)$$

It is easy to prove that

i) Ξ is a bimodule over A . (A proof of this first statement as well as of the following ones is contained in [2]). □

Introduce a left action (push-forward) of the Hopf algebra A on Ξ

Definition

$$\Delta_{\Xi}(a^i t_i) \equiv \Delta(a^i)(I \otimes t_i) . \quad (4.5)$$

It follows that

ii) (Ξ, Δ_{Ξ}) is a left covariant bimodule over A , that is

$$\Delta_{\Xi}(aVb) = \Delta(a)\Delta_{\Xi}(V)\Delta(b) ; \quad (\varepsilon \otimes id)\Delta_{\Xi}(V) = V ; \quad (\Delta \otimes id)\Delta_{\Xi} = (id \otimes \Delta_{\Xi})\Delta_{\Xi} .$$

□

Introduce n^2 elements of A , N_i^j , satisfying [see (2.33), (2.29) and (2.30)]

$$N_i^j(a * O_i^k) = (O_j^i * a)N_i^k \quad (4.6)$$

$$\Delta(N_i^j) = N_i^j \otimes I \quad (4.7)$$

$$\varepsilon(N_i^j) = \delta_j^i , \quad (4.8)$$

and introduce a right action $\Xi \Delta$ such that [see (2.28)]

Definition

$$\Xi \Delta(a^i t_i) \equiv \Delta(a^i)(t_j \otimes N_i^j) . \quad (4.9)$$

Then it can be proven that

iii) The elements [see (2.32)]

$$h_i \equiv S^{-1}(N_i^j)t_j \quad (4.10)$$

are right invariant: $\Xi \Delta(h_i) = h_i \otimes I$. Moreover any $V \in \Xi$ can be expressed in a unique way respectively as $V = a^i h_i$ and as $V = h_i b^i$, where $a^i, b^i \in A$.

□

iv) $(\Xi, {}_{\Xi}\Delta)$ is a right covariant bimodule over A , that is

$${}_{\Xi}\Delta(aVb) = \Delta(a){}_{\Xi}\Delta(V)\Delta(b) ; \quad (id \otimes \varepsilon)_{\Xi}\Delta(V) = V ; \quad (id \otimes \Delta)_{\Xi}\Delta = ({}_{\Xi}\Delta \otimes id)_{\Xi}\Delta .$$

□

v) The left and right covariant bimodule $(\Xi, \Delta_{\Xi}, {}_{\Xi}\Delta)$ is a bicovariant bimodule, that is left and right actions are compatible:

$$(id \otimes {}_{\Xi}\Delta)\Delta_{\Xi} = (\Delta_{\Xi} \otimes id)_{\Xi}\Delta .$$

□□

In the previous section we have seen [remark (II)] that the space of vector fields Ξ is the free left A -module generated by the symbols t_i , so that the above theorem applies to our case.

There are many bimodule structures (i.e. choices of O_i^j) Ξ can be endowed with. Using the fact that Ξ is dual to Γ we request compatibility with the Γ bimodule. In the commutative case $\langle ft_j, \omega^i \rangle = \langle t_{j\Box} f, \omega^i \rangle = \langle t_j, f\omega^i \rangle = \langle t_j, \omega^i f \rangle$. In the quantum case we know that $\langle t_j, \omega^i a \rangle = \langle at_j, \omega^i \rangle$ and we require

$$\langle t_{j\Box} a, \omega^i \rangle = \langle t_j, a\omega^i \rangle ; \quad (4.11)$$

this condition uniquely determines the bimodule structure of Ξ . Indeed we have

$$t_{i\Box} a = (f_i^j * a) t_j \quad (4.12)$$

i.e.

$$O_i^j \equiv f_i^j . \quad (4.13)$$

This is in fact the bimodule structure needed for the product rule (3.8). So far Ξ has a bimodule structure. Ξ becomes a left covariant bimodule if we define Δ_{Ξ} as in (4.5) so that t_i are left invariant vector fields. The following consideration fixes the right covariance of Ξ : For reasons of symmetry let us try to impose

$$\langle t_i, \omega^j \rangle = \delta_i^j = \langle h_i, \eta^j \rangle, \quad (4.14)$$

where t_i and ω^j are left-invariant and h_i and η^j are the canonically associated right-invariant objects; see (4.10) and (2.32). As can easily be seen this condition is equivalent to the following definition of the n^2 elements $N_k^l \in A$

Definition

$$N_k^l = S(M_k^l) . \quad (4.15)$$

Theorem 4. The N_k^l elements defined above satisfy relations (4.8), (4.7) and (4.6):

$$1) \varepsilon(N_i^j) = \delta_j^i \quad 2) \Delta(N_i^j) = N_i^j \otimes 1 \quad 3) N_k^i(a * O_i^j) = (O_k^i * a) N_i^j$$

Proof

1) is trivial.

2) use $N^i_j = SM_j^i$ and $\Delta \circ S = \sigma \circ (S \otimes S) \circ \Delta$, where σ_A is the flip map in $A \otimes A$.

3) Using $N^i_j = SM_j^i$ and $O_i^j = f_i^j$ in (2.33) gives

$$(a * O_k^i)S^{-1}N^j_i = S^{-1}N^i_k(O_i^j * a).$$

Multiplying with “ N ” on both sides we obtain relation 3).

□□□

In equations (2.2) and (2.16) elements of Ξ and Γ make a joint appearance. To be still able to talk about transformation properties of such expressions we need to combine the previously introduced actions into one object, Δ_A , simply by putting $\Delta_A \equiv \Xi\Delta$ on Ξ and $\Delta_A \equiv \Gamma\Delta$ on Γ and requiring Δ_A to be an algebra homomorphism. From this definition we get the following important corollary:

Corollary. The expression $\omega^i t_i$ in (2.2) is invariant in the sense that

$$\Delta_A(\omega^i t_i) = \Gamma\Delta(\omega^i)_{\Xi}\Delta(t_i) = \omega^j t_k \otimes M_j^i N^k_i = \omega^i t_i \otimes 1.$$

Similar statements apply to ${}_A\Delta$.

The left-invariant vector fields were characterized in (2.9) through their left action $t_i(a) = \chi_i * a$ on functions. It would be nice if a similar equation were true for the right-invariant vector fields h_i ; indeed we have

Theorem 5. $h_i(a) \equiv S^{-1}(N^j_i)t_j(a) = a * \chi_i \quad \forall a \in A.$

Proof

From (2.2) we have

$$\omega_a \equiv Pda \equiv d(a_2)S^{-1}a_1 = \omega^i \chi_i(a)$$

and in particular $\omega_{x^i} = \omega^i$. Using this equation we can rewrite the definition of M_i^k , (2.28), as $M_i^k = \chi_i(x_2^k)x_3^k S^{-1}x_1^k$, or

$$M_i^j \chi_j(x^k) = \chi_i(x_2^k)x_3^k S^{-1}x_1^k.$$

We now show that this equation is still true for an arbitrary function $a \in A$ in place of x^k . Indeed from (2.3) any function can be decomposed into a part that is spanned by the x^i and a part that is contained in $R \oplus \{I\}$ i.e. is annihilated by the χ_j . But R and $\{I\}$ are Ad-invariant so the part of a that is contained in $R \oplus \{I\}$ contributes to neither side of the equation in question and we have in fact:

$$M_i^j \chi_j(a) = \chi_i(a_2)a_3 S^{-1}a_1.$$

Definition (4.15) and a slight rearrangement (take $a = b_2$ and multiply by b_1) finally gives

$$S^{-1}N^j_i a_1 \chi_j(a_2) = \chi_i(a_1)a_2$$

and proves the theorem. □□□

Remark. Statement 3) of Theorem 4 is in fact a consequence of Theorem 5. Both are a consequence of a more general Hopf algebra relation (*e.g.* eqn. 4.67 of [14]) that is valid in the semi-direct product algebra of A and A^* .

Now that we have all the ingredients, the construction of the bicovariant bimodule Ξ is easy and straightforward. For example $\Xi\Delta$ is given in formula (4.9).

We can then conclude that $(\Xi, \Delta_\Xi, \Xi\Delta)$ is a bicovariant bimodule.

Notice that, since Theorem 3 completely characterizes a bicovariant bimodule all the formulas containing the symbols f_i^j or M_k^l or elements of Γ are still valid under the substitutions $f_i^j \rightarrow S^{-1}O_i^j$, $M_k^l \rightarrow N_k^l$ and $\Gamma \rightarrow \Xi$.

5 Tensor fields

The construction completed for vector fields is readily generalized to p -times contravariant tensor fields.

We define $\Xi \otimes \Xi$ to be the space of all elements that can be written as finite sums of the kind $\sum_i V_i \otimes V'_i$ with $V_i, V'_i \in \Xi$. The tensor product (in the algebra A) between V_i and V'_i has the following properties:

$$V \square a \otimes V' = V \otimes aV', \quad a(V \otimes V') = (aV) \otimes V' \text{ and } (V \otimes V') \square a = V \otimes (V' \square a) \quad (5.1)$$

so that $\Xi \otimes \Xi$ is naturally a bimodule over A .

Left and right actions on $\Xi \otimes \Xi$ are defined by:

$${}_A\Delta(V \otimes V') \equiv V_1 V'_1 \otimes V_l \otimes V'_l, \quad {}_A\Delta : \Xi \otimes \Xi \rightarrow A \otimes \Xi \otimes \Xi \quad (5.2)$$

$$\Delta_A(V \otimes V') \equiv V_r \otimes V'_r \otimes V_1 V'_1, \quad \Delta_A : \Xi \otimes \Xi \rightarrow \Xi \otimes \Xi \otimes A \quad (5.3)$$

where $V_1 \otimes V_l$ and $V_r \otimes V_1$ are defined by

$$\Delta_\Xi(V) = V_1 \otimes V_l \in A \otimes \Xi, \quad (5.4)$$

$$\Xi\Delta(V) = V_r \otimes V_1 \in \Xi \otimes A. \quad (5.5)$$

More generally, we can introduce the actions on $\Xi^{\otimes p} \equiv \underbrace{\Xi \otimes \Xi \otimes \cdots \otimes \Xi}_{p\text{-times}}$ as

$${}_A\Delta(V \otimes V' \otimes \cdots \otimes V'') \equiv V_1 V'_1 \cdots V''_1 \otimes V_l \otimes V'_l \otimes \cdots \otimes V''_l$$

$${}_A\Delta : \Xi^{\otimes p} \longrightarrow A \otimes \Xi^{\otimes p} ; \quad (5.6)$$

$$\Delta_A(V \otimes V' \otimes \cdots \otimes V'') \equiv V_r \otimes V'_r \otimes \cdots \otimes V''_r \otimes V_1 V'_1 \cdots V''_1$$

$$\Delta_A : \Xi^{\otimes p} \longrightarrow \Xi^{\otimes p} \otimes A . \quad (5.7)$$

Left invariance on $\Xi \otimes \Xi$ is naturally defined as ${}_A\Delta(V \otimes V') = I \otimes V \otimes V'$ (similar definition for right invariance), so that for example $t_i \otimes t_j$ is left invariant, and is in fact a left invariant basis for $\Xi \otimes \Xi$: each element can be written as $a^{ij} t_i \otimes t_j$ in a unique way.

It is not difficult to show that $\Xi \otimes \Xi$ is a bicovariant bimodule. In the same way also $(\Xi^{\otimes p}, {}_A\Delta, \Delta_A)$ is a bicovariant bimodule. An analog procedure, using Δ_Γ and ${}_r\Delta$ instead of Δ_Ξ and $\Xi\Delta$, applies also to $\Gamma^{\otimes n}$ the n -times tensor product of 1-forms.

Any element $v \in \Xi^{\otimes p}$ can be written as $v = b^{i_1 \dots i_p} t_{i_1} \otimes \dots \otimes t_{i_p}$ in a unique way, similarly any element $\tau \in \Gamma^{\otimes n}$ can be written as $\tau = \omega^{i_1} \otimes \dots \otimes \omega^{i_n} a_{i_1 \dots i_n}$ in a unique way. It is now possible to generalize the previous bracket $\langle \ , \ \rangle : \Xi \times \Gamma \rightarrow A$ to $\Xi^{\otimes p}$ and $\Gamma^{\otimes n}$. For $p > n$: $\langle v, \tau \rangle = 0$; for $p \leq n$:

$$\begin{aligned} \langle \ , \ \rangle : \quad \Xi^{\otimes p} \times \Gamma^{\otimes n} &\longrightarrow \Gamma^{\otimes n-p} \\ (v, \tau) &\longmapsto \langle v, \tau \rangle = b^{j_1 \dots j_p} \langle t_{j_1} \otimes \dots \otimes t_{j_p}, \omega^{i_1} \otimes \dots \otimes \omega^{i_n} \rangle a_{i_1 \dots i_n} \\ &= b^{i_p \dots i_1} \omega^{i_{p+1}} \otimes \dots \otimes \omega^{i_n} a_{i_1 \dots i_n} \end{aligned} \quad (5.8)$$

where $\Gamma^{\otimes 0} \equiv A$, $\Gamma^{\otimes 1} \equiv \Gamma$ and we have defined

$$\begin{aligned} \langle t_{j_p} \otimes \dots \otimes t_{j_1}, \omega^{i_1} \otimes \dots \otimes \omega^{i_n} \rangle &\equiv \langle t_{j_p} \otimes \dots \otimes t_{j_1}, \omega^{i_1} \otimes \dots \otimes \omega^{i_p} \rangle \omega^{i_{p+1}} \otimes \dots \otimes \omega^{i_n} \\ &\equiv \langle t_{j_1}, \omega^{i_1} \rangle \dots \langle t_{j_p}, \omega^{i_p} \rangle \omega^{i_{p+1}} \otimes \dots \otimes \omega^{i_n} \\ &= \delta_{j_1}^{i_1} \dots \delta_{j_p}^{i_p} \omega^{i_{p+1}} \otimes \dots \otimes \omega^{i_n} \end{aligned} \quad (5.9)$$

(with the obvious convention for the special cases $p = n - 1$ and $p = n$). Using definition (5.9) it is easy to prove that

$$\langle v \sqcup a, \tau \rangle = \langle v, a\tau \rangle , \quad (5.10)$$

namely

$$\begin{aligned} \langle t_{j_1} \otimes \dots \otimes t_{j_p} \sqcup a, \omega^{i_1} \otimes \dots \otimes \omega^{i_n} \rangle &= (f_{j_1}^{k_1} * \dots * f_{j_p}^{k_p} * a) \langle t_{k_1} \otimes \dots \otimes t_{k_p}, \omega^{i_1} \otimes \dots \otimes \omega^{i_n} \rangle \\ &= (f_{j_1}^{i_p} * \dots * f_{j_p}^{i_1} * a) \omega^{i_{p+1}} \otimes \dots \otimes \omega^{i_n} \\ \langle t_{j_1} \otimes \dots \otimes t_{j_p}, a\omega^{i_1} \otimes \dots \otimes \omega^{i_n} \rangle &= \langle t_{j_1} \otimes \dots \otimes t_{j_p}, \omega^{l_1} \otimes \dots \otimes \omega^{l_n} \rangle (f_{l_n}^{i_n} * \dots * f_{l_1}^{i_1} * a) \\ &= \delta_{j_1}^{l_1} \dots \delta_{j_p}^{l_p} \omega^{l_{p+1}} \otimes \dots \otimes \omega^{l_n} (f_{l_n}^{i_n} * \dots * f_{l_1}^{i_1} * a) \\ &= (f_{j_1}^{i_p} * \dots * f_{j_p}^{i_1} * a) \omega^{i_{p+1}} \otimes \dots \otimes \omega^{i_n} \end{aligned}$$

and (5.10) is verified if and *only if* (5.9) holds.

Therefore we have also shown that definition (5.9) is the only one compatible with property (5.10), i.e. property (5.10) uniquely determines the coupling between Ξ^\otimes and Γ^\otimes .

It is also easy to prove that the bracket $\langle \cdot, \cdot \rangle$ extends to $\Xi^{\otimes p}$ and $\Gamma^{\otimes p}$ the duality between Ξ and Γ .

More generally we can define $\Xi^\otimes \equiv A \oplus \Xi \oplus \Xi^{\otimes 2} \oplus \Xi^{\otimes 3} \dots$ to be the algebra of contravariant tensor fields (and Γ^\otimes that of covariant tensor fields).

The actions ${}_A\Delta$ and Δ_A have a natural generalization to Ξ^\otimes so that we can conclude that $(\Xi^\otimes, {}_A\Delta, \Delta_A)$ is a bicovariant graded algebra, the graded algebra of tensor fields over the ring “of functions on the group” A , with the left and right “push-forward” ${}_A\Delta$ and Δ_A .

Before we can introduce a Lie derivative and an inner derivation we need a wedge product of forms, *i.e.* we have to briefly discuss antisymmetrized covariant tensor fields.

5.1 Bicovariant graded algebras

A general method to construct new tensor fields starting from Γ^\otimes is to quotient the bicovariant graded algebra Γ^\otimes with a graded ideal S . In order for Γ^\otimes/S to be bicovariant, the ideal S has also to be biinvariant (i.e. a left and right sub-comodule). For example the ideal can be generated by the elements $S_{ij}^{kl}\omega^i \otimes \omega^j$ ⁶ and a sufficient condition for the biinvariance is $S_{ij}^{kl}M_k^m M_l^n = M_i^k M_j^l S_{kl}^{mn}$, where $\Xi\Delta\omega^i = \omega^j \otimes M_j^i$.

Particular cases of such a quotient procedure are when S is the kernel of a graded bimodule homomorphism $A : \Gamma^\otimes \rightarrow \Gamma^\otimes$ then $\Gamma^\otimes/\ker(A)$ is isomorphic to the image of A and therefore is a biinvariant graded subalgebra of Γ^\otimes . This is what we require for a wedge product construction. The wedge product of two forms is given by

$$\begin{aligned} \omega^i \wedge \omega^j &= A(\omega^i \otimes \omega^j) = \omega^k \otimes \omega^l W_{kl}^{ij} \in \Gamma^{\wedge 2} \subset \Gamma^{\otimes 2} \\ a\omega^i \wedge \omega^j &\equiv a(\omega^i \wedge \omega^j) ; \quad \omega^i \wedge \omega^j b \equiv (\omega^i \wedge \omega^j)b \end{aligned} \quad (5.11)$$

Since the homomorphism A is required to commute with the coactions ${}_A\Delta$ and Δ_A the new tensor transforms covariantly according to its index structure:

$$\Delta_A(\omega^i \wedge \omega^j) \equiv \Delta_A(\omega^k \otimes \omega^l) W_{kl}^{ij} = \omega^k \wedge \omega^l \otimes M_k^i M_l^j \quad (5.12)$$

thus imposing

$$W_{ij}^{kl} M_k^m M_l^n = M_i^k M_j^l W_{kl}^{mn} \quad (5.13)$$

on W as a necessary condition. In short tensor product notation: $\omega_1 \wedge \omega_2 = \omega_1 \otimes \omega_2 W_{12} \Rightarrow W_{12} M_1 M_2 = M_1 M_2 W_{12}$. Let $\hat{\sigma}_{ij}^{kl} = f_i^l(M_j^k)$. Due to (2.33) any function $W(1, \hat{\sigma})$ will satisfy (5.13). We will in fact take $W = 1 - \hat{\sigma}$. For this choice and for examples of $n > 2$ relations we would like to refer the reader to [2, 7, 9, 14];

⁶Or, what is the same, one imposes relations $S_{ij}^{kl}\omega^i \wedge \omega^j = 0$.

here we will just quote the results: generalizing equation (5.11), wedge products of n forms are again expressed in terms of tensor fields:

$$\omega_1 \wedge \dots \wedge \omega_n = \omega_1 \otimes \dots \otimes \omega_n W_{1\dots n}. \quad (5.14)$$

The numerical coefficients $W_{1\dots n}$ are given through a recursion relation

$$W_{1\dots n} = W_{2\dots n} \mathcal{I}_{1\dots n}, \quad (5.15)$$

where

$$\mathcal{I}_{1\dots n} = 1 - \hat{\sigma}_{12} + \hat{\sigma}_{12}\hat{\sigma}_{23} - \dots - (-1)^n \hat{\sigma}_{12}\hat{\sigma}_{23} \dots \hat{\sigma}_{n-1,n} \quad (5.16)$$

and $W_i^j = \mathcal{I}_i^j = \delta_i^j$. As is easily seen, \mathcal{I} has the following decomposition property that we will use in the next chapter:

$$\mathcal{I}_{1\dots n} = \mathcal{I}_{1\dots k} + (-1)^k \hat{\sigma}_{12}\hat{\sigma}_{23} \dots \hat{\sigma}_{k,k+1} \mathcal{I}_{k+1\dots n}. \quad (5.17)$$

The first order differential operator d can be extended in a unique way to an exterior differential d mapping n -forms in $(n+1)$ -forms, satisfying the undeformed Leibniz rule and such that $d^2 = 0$ [2].

It is also straightforward to define antisymmetrized contravariant vector fields in this way.

6 Contraction operator and Lie derivative

In this last section we propose a definition of Lie derivative along a generic vector field. We start with the introduction of the contraction operator i_V , also called inner derivation, with $V \in \Xi$ based on equation (5.8).

For a generic vector field $V = b^j t_j$ we define:

Definition

$$i_V(\vartheta) = \langle V, \vartheta \rangle \quad \vartheta \text{ generic form.}$$

Theorem 6. The contraction operator satisfies the following properties:

ϑ, ϑ' generic forms; $a, a_{i_1\dots i_n} \in A$; $V = b^i t_i$; $\lambda \in \mathbf{C}$

- a) $i_V(\vartheta) = i_{b^j t_j}(\vartheta) = b^j i_{t_j}(\vartheta)$
- b) $i_V(a) = 0$
- c) $i_V(\omega^j) = b^j$
- d) $i_V(\omega^{i_1} \wedge \dots \wedge \omega^{i_n} a_{i_1\dots i_n}) = i_V(\omega^{i_1} \wedge \dots \wedge \omega^{i_s}) \wedge (\omega^{i_{s+1}} \wedge \dots \wedge \omega^{i_n} a_{i_1\dots i_n}) +$
 $(-1)^s b^i f_i^j * (\omega^{i_1} \wedge \dots \wedge \omega^{i_s}) \wedge i_{t_j}(\omega^{i_{s+1}} \wedge \dots \wedge \omega^{i_n} a_{i_1\dots i_n})$

- e) $i_V(a\vartheta) = b^i(f_i^j * a) i_{t_j}(\vartheta)$
 f) $i_V(\vartheta a + \vartheta') = i_V(\vartheta)a + i_V(\vartheta')$
 g) $i_{\lambda V} = \lambda i_V$

Proof.

Properties a), b), c), f), g) are direct consequences of (5.8), e) follows from (5.10). To proof d) we have to use the definition of the wedge product (5.14 – 5.16): first we note that (in tensor product notation)

$$\begin{aligned} i_V(\omega_1 \wedge \dots \wedge \omega_n) &= i_V(\omega_1 \otimes \dots \otimes \omega_n W_{1\dots n}) \\ &= i_V(\omega_1) \omega_2 \otimes \dots \otimes \omega_n W_{1\dots n} \\ &= i_V(\omega_1) \omega_2 \wedge \dots \wedge \omega_n \mathcal{I}_{1\dots n}, \end{aligned}$$

where we have used (5.15) in the last step — in index notation:

$$i_{t_i}(\omega^{i_1} \wedge \dots \wedge \omega^{i_n}) = \omega^{j_2} \wedge \dots \wedge \omega^{j_n} \mathcal{I}_{ij_2\dots j_n}^{i_1\dots i_n};$$

next we can show

$$\begin{aligned} f_i^j * (\omega^{i_1} \wedge \dots \wedge \omega^{i_s}) &= \omega^{k_1} \wedge \dots \wedge \omega^{k_s} f_i^j (M_{k_1}^{i_1} \dots M_{k_s}^{i_s}) \\ &= \omega^{k_2} \wedge \dots \wedge \omega^{k_{s+1}} \hat{\sigma}_{ik_2}^{i_1 l_2} \hat{\sigma}_{l_2 k_3}^{i_2 l_3} \dots \hat{\sigma}_{l_s k_{s+1}}^{i_s j}, \end{aligned}$$

finally we utilize the decomposition property (5.17) and associativity of the wedge product (in tensor product notation)

$$\begin{aligned} i_V(\omega_1 \wedge \dots \wedge \omega_n) &= \\ &= i_V(\omega_1) \omega_2 \wedge \dots \wedge \omega_n \mathcal{I}_{1\dots n} \\ &= i_V(\omega_1) (\omega_2 \wedge \dots \wedge \omega_s) \wedge (\omega_{s+1} \wedge \dots \wedge \omega_n) (\mathcal{I}_{1\dots s} + (-1)^s \hat{\sigma}_{12} \dots \hat{\sigma}_{s, s+1} \mathcal{I}_{s+1\dots n}) \\ &= i_V(\omega_1 \wedge \dots \wedge \omega_s) \wedge (\omega_{s+1} \wedge \dots \wedge \omega_n) \\ &\quad + b^i i_{t_i}(\omega_1) (\omega_2 \wedge \dots \wedge \omega_{s+1}) \wedge (\omega_{s+2} \wedge \dots \wedge \omega_n) (-1)^s \hat{\sigma}_{12} \dots \hat{\sigma}_{s, s+1} \mathcal{I}_{s+1\dots n} \\ &= i_V(\omega_1 \wedge \dots \wedge \omega_s) \wedge \omega_{s+1} \wedge \dots \wedge \omega_n \\ &\quad + (-1)^s b^i f_i^j * (\omega_1 \wedge \dots \wedge \omega_s) \wedge i_{t_j}(\omega_{s+1} \wedge \dots \wedge \omega_n). \end{aligned}$$

With property f) this proves d). □□□

Remark 1. A slight generalization of property d) for two generic forms ϑ and ϑ' is also true [use e)]:

$$i_V(\vartheta \wedge \vartheta') = i_V \vartheta \wedge \vartheta' + (-1)^{\deg(\vartheta)} b^i f_i^j * \vartheta \wedge i_{t_j} \vartheta'. \quad (6.1)$$

Remark 2. Property e) reduces in the commutative case to the familiar formula:

$$i_V(h\vartheta) = hi_V\vartheta .$$

The Lie derivative along left invariant vector fields is given by [7]:

$$\ell_t(\tau) \equiv (id \otimes \chi)_\Gamma \Delta(\tau) = \chi * \tau \quad \ell_t : \Gamma^{\otimes n} \longrightarrow \Gamma^{\otimes n} \quad (6.2)$$

where $\chi \in {}_{inv}\Xi$ is such that $\chi * = t$. On Γ^\wedge it can be proven that [6, 7, 15] :

$$\ell_{t_i} = i_{t_i}d + di_{t_i} . \quad (6.3)$$

It is then natural to introduce the Lie derivative along a generic vector field V through the following

Definition

$$\ell_V = i_Vd + di_V \quad (6.4)$$

Theorem 7. The Lie derivative satisfies the following properties:

- 1) $\ell_V a = V(a)$
- 2) $\ell_V d\vartheta = d\ell_V\vartheta$
- 3) $\ell_V(\lambda\vartheta + \vartheta') = \lambda\ell_V\vartheta + \ell_V\vartheta'$
- 4) $\ell_{bV}\vartheta = b\ell_V\vartheta + db \wedge i_V\vartheta$, where ϑ is a generic form
- 5) $\ell_V(\vartheta \wedge \vartheta') = \ell_V\vartheta \wedge \vartheta' + b^i(f_i^j * \vartheta) \wedge \ell_{t_j}\vartheta' + (-1)^{\deg(\vartheta)} db^i \wedge (f_i^j * \vartheta) \wedge i_{t_j}(\vartheta')$,
where ϑ, ϑ' are generic forms and $V = b^i i_{t_i}$.

Proof.

Properties 1), 2), 3) and 4) follow directly from the definition (6.4).

Property 5) is also a consequence of definition (6.4); the proof makes use of (6.1) and $d(f_j^k * \vartheta) = f_j^k * d\vartheta$.

□□□

7 Commutation relations

The careful choice of conventions (see introduction) allows that all relations can be summarized as deformed commutation relations [9] of right-acting operators (ϑ : p -form, $V = b^i t_i$)

$$d\vartheta = d(\vartheta) + (-1)^p \vartheta d \quad (7.5)$$

$$i_V\vartheta = i_V(\vartheta) + (-1)^p b^i (f_i^j * \vartheta) i_{t_j} \quad (7.6)$$

$$\ell_V\vartheta = \ell_V(\vartheta) + b^i (f_i^j * \vartheta) \ell_{t_j} + (-1)^p db^i (f_i^j * \vartheta) i_{t_j} \quad (7.7)$$

and their actions ($a \in A$, rest as above) as given by:

$$i_V(a) = 0 ; \quad i_V(\omega^j) = b^j ; \quad \ell_V(a) = V(a) = b^i \chi_i * a ; \quad \ell_V \omega^j = b^i \chi_i * \omega^j + db^j . \quad (7.8)$$

These operators form a graded quantum Lie algebra

$$\{d, d\} = 0 \quad (7.9)$$

$$[d, \ell_V] = 0 \quad (7.10)$$

$$\{d, i_V\} = \ell_V \quad (7.11)$$

which is supplemented by two more relations⁷

$$[\ell_{t_i}, \ell_{t_k}]_{\hat{B}} = \ell_{[t_i, t_k]_{\hat{B}}} = f_i^l{}_k \ell_{t_l} \quad (7.12)$$

$$[\ell_{t_i}, i_{t_k}]_{\hat{B}} = i_{[t_i, t_k]_{\hat{B}}} = f_i^l{}_k i_{t_l} \quad (7.13)$$

where $\hat{B}_{ik}^{rs} \equiv (\widehat{\sigma^{-1}})_{ik}^{rs} = f_i^s(N^r{}_k)$, $[\ell_{t_i}, \ell_{t_k}]_{\hat{B}} \equiv \ell_{t_i} \ell_{t_k} - \hat{B}_{ik}^{rs} \ell_{t_r} \ell_{t_s}$ and $f_i^l{}_k = \chi_i(N^l{}_k)$.⁸

Using the right product (4.12) the commutation relations can be restated in basis-independent form as

$$V\vartheta = V(\vartheta) + V_{\square}\vartheta. \quad (7.14)$$

Here and in the following V stands for any of the first-order operators $V = b^i t_i$ (for $\vartheta \in A$ only) or i_V , ℓ_{t_i} and d . For example:

$$t_i a = t_i(a) + t_{i\square} a = \chi * a + (f_i^j * a) t_j ; \quad d\vartheta = d(\vartheta) + d_{\square}\vartheta = d(\vartheta) + (-1)^{\deg(\vartheta)} \vartheta d .$$

For ϑ a p -form one must add the following rules to the definition (4.12) of the right product to take care of the grading:

$$d_{\square}\vartheta = (-1)^p \vartheta d ; \quad i_{V\square}\vartheta = (-1)^p b^i (f_i^j * \vartheta) i_{t_j} . \quad (7.15)$$

Note that V appears twice in the r.h.s. of (7.14). These are the two parts of the (classical) coproduct

$$\delta(V) = V \otimes 1 + 1 \otimes V. \quad (7.16)$$

In a hopefully self-explanatory notation we rewrite (7.14) as

$$V\vartheta = \delta(V)_{\square}\vartheta \quad (7.17)$$

so that the role of $\delta(V)$ is emphasized.

We now extend the definition of δ to composition of operators, say $V \cdot V' \cdots V''$:

$$\forall \vartheta \in \Gamma^{\wedge} \quad \delta(V \cdot V' \cdots V'')_{\square}\vartheta \equiv V \cdot V' \cdots V'' \vartheta . \quad (7.18)$$

⁷Relations for the inner derivations can be constructed along the lines of section 5.1 .

⁸In these notations the Cartan-Maurer formula reads:

$d\omega^i \otimes \chi_i = \omega^i \otimes \omega^j \otimes \chi_j \chi_i = -\omega^a \wedge \omega^b \hat{\sigma}_{ab}^{pq} [\chi_q, \chi_p]_{\hat{B}}$. In particular $d\omega^i = \omega^a \otimes \omega^b \chi_a (M_b^i)$.

From equation (2.33) one can prove that

$$f_i^k * [i_{t_j}(\vartheta)] = \hat{B}_{ij}^{rs} i_{t_r} [f_s^k * \vartheta] \quad (7.19)$$

the following theorem is then easily derived:

Theorem 8.

The operators i_V , ℓ_{t_i} , d and V ($V = b^i t_i$ is only defined on A) form a braided tensor algebra. The braidings are given by:

$$\begin{aligned} (1 \otimes a^i i_{t_i})(b^j i_{t_j} \otimes 1) &= -a^i (f_i^h * b^j) \hat{B}_{hj}^{rs} (i_{t_r} \otimes i_{t_s}) ; \quad (1 \otimes d)(d \otimes 1) = -d \otimes d \\ (1 \otimes d)(i_V \otimes 1) &= -i_V \otimes d ; \quad (1 \otimes i_V)(d \otimes 1) = -d \otimes i_V . \end{aligned} \quad (7.20)$$

Using (6.3) we also get all the other braidings, e.g.

$$(1 \otimes a^i \ell_{t_i})(b^j \ell_{t_j} \otimes 1) = a^i (f_i^h * b^j) \hat{B}_{hj}^{rs} (\ell_{t_r} \otimes \ell_{t_s}). \quad (7.21)$$

From definition (7.18) and the above theorem it follows:

Theorem 9.

δ is a homomorphism in the *braided* tensor algebra of operators.

For example:

$$\delta(t_i t_j) = \delta(t_i) \delta(t_j) \quad (7.22)$$

indeed

$$\begin{aligned} \delta(t_i t_j) \lrcorner a &\equiv t_i t_j a = t_i t_j (a) + t_i (f_j^l * a) t_l \\ &= t_i (t_j(a)) + (f_i^l * t_j(a)) t_l + [t_i (f_j^l * a)] t_l + (f_i^m * f_j^l * a) t_m t_l \\ &= (t_i t_j \otimes 1 + (1 + \hat{B})_{ij}^{rs} (t_r \otimes t_s) + 1 \otimes t_i t_j) \lrcorner a \\ &= (t_i \otimes 1 + 1 \otimes t_i) (t_j \otimes 1 + 1 \otimes t_j) \lrcorner a \\ &= [\delta(t_i) \delta(t_j)] \lrcorner a \end{aligned} \quad (7.23)$$

where we have used $\hat{B}_{ij}^{rs} \chi_r * f_s^l = f_i^l * \chi_j$ an identity easily derived from (6.3) and (7.19), or directly from $d(f_i^j * a) = f_i^j * da$.

As another example we have:

$$\delta(\ell_V) = \delta i_V \delta d + \delta d \delta i_V$$

indeed use the braiding (7.20) to show that $(\delta i_V \delta d + \delta d \delta i_V) \lrcorner \vartheta$ is given by (7.7).

To summarize: we can state commutation relations for a general differential operator D and for any form ϑ in an abstract way as

$$D\vartheta = \delta(D) \lrcorner \vartheta, \quad (7.24)$$

where δ is the *classical* coproduct on first order operators and can be extended (Theorem 9) as a homomorphism to higher order operators.

8 A second Lie derivative and a puzzle

It is natural to define a Lie derivative $\ell_{h_i}^{\mathcal{R}}$ of a generic form τ along a *right*-invariant vector field h_i in terms of the *left* coaction Δ_Γ :

$$\ell_h^{\mathcal{R}}(\tau) \equiv (\chi \otimes id)\Delta_\Gamma(\tau) = \tau * \chi ,$$

just like it was natural that we used the *right* coaction, when we defined ℓ_{t_i} in the previous section. In this section we would like to compare the two definitions.

From the above definition we find

$$\begin{aligned} \ell_{h_i}^{\mathcal{R}}(\vartheta\vartheta') &= \chi(\vartheta_1\vartheta'_1)\vartheta_l\vartheta'_l \\ &= \chi(\vartheta_1)\vartheta_l\vartheta' + f_i^j(\vartheta_1)\chi_j(\vartheta'_1)\vartheta_l\vartheta'_l \\ &= \ell_{h_i}^{\mathcal{R}}(\vartheta)\vartheta' + (\vartheta * f_i^j)\ell_{h_j}^{\mathcal{R}}(\vartheta') \end{aligned} \quad (8.25)$$

and, in a particular case:

$$\begin{aligned} \ell_{h_i}^{\mathcal{R}}(da\,b) &= d(h_i(a))b + (da * f_i^j)h_j(b) \\ &= d(h_i(a))b + (da * f_i^j)S^{-1}N^k{}_j t_k(b). \end{aligned} \quad (8.26)$$

On the other hand we can use the results of the previous chapter to give an alternative expression for the Lie derivative along the right-invariant vector field h_i :

$$\begin{aligned} \ell_{h_i} &= S^{-1}N^j{}_i \ell_{t_j} + d(S^{-1}N^j{}_i) \wedge i_{t_j} \\ &= M_i^j \ell_{t_j} + dM_i^j \wedge i_{t_j} \end{aligned} \quad (8.27)$$

and hence

$$\begin{aligned} \ell_{h_i}(da\,b) &= d(M_i^j t_j(a))b + M_i^j(f_j^k * da)t_k(b) \\ &= d(h_i(a))b + M_i^j(f_j^k * da)t_k(b). \end{aligned} \quad (8.28)$$

The difference between expressions (8.26) and (8.28) is a good index for the “defect” between left and right transports on a quantum group:

$$\begin{aligned} (\ell_{h_i} - \ell_{h_i}^{\mathcal{R}})(da\,b) &= [M_i^j(f_j^k * da) - (da * f_i^j)M_j^k]t_k(b) \\ &= -DI_i^k(a)t_k(b); \end{aligned} \quad (8.29)$$

$$DI_i^k(a) \equiv [d(M_i^j)(f_j^k * a) - (a * f_i^j)d(M_j^k)] \quad (\text{Defect Index}). \quad (8.30)$$

In the last passage we have used the Leibniz rule for d combined with (2.33). The term in the square brackets is always zero in the classical (undeformed) case. Note that $(\ell_{h_i} - \ell_{h_i}^{\mathcal{R}})$ vanishes on da and b separately but not necessarily on $da\,b$. The case of “ b ” confirms a result from Section 4:

$$\ell_{h_i}(b) = S^{-1}N^j{}_i t_j(b) = h_i(b) = \ell_{h_i}^{\mathcal{R}}(b) \quad (8.31)$$

and shows that we will not encounter any ambiguities or inconsistencies as long as we deal with general vector fields and functions alone. Problems can occur however when we start to introduce forms. Here is what happens in a well known example:

In the approach [3, 13] to differential calculus on quantum groups that is based on the L^\pm and T matrices of [1] one can give the following explicit expressions for the f_i^j and the linearly independent χ_k functionals (using double-indices)

$$f_{i_1 i_2}^{j_1 j_2} = L^{+i_1}_{j_1} S L^{-j_2}_{i_2} = \varepsilon \delta_{k_2}^{k_1} - L^{+k_1}_j S L^{-j}_{k_2}$$

and for the adjoint representation N^j_i, M_i^j

$$N^{j_1 j_2}_{i_1 i_2} = S T^{i_1}_{j_1} T^{j_2}_{i_2} \quad , \quad M_{i_1 i_2}^{j_1 j_2} = (S^{-1} T^{j_2}_{i_2}) T^{i_1}_{j_1} \quad .$$

We want to investigate whether the “Defect Index” vanishes. First consider the $x^{a_1}_{a_2}$ of (2.4) such that $\chi^{k_1}_{k_2}(x^{a_1}_{a_2}) = \delta^{k_1 a_1} \delta_{k_2 a_2}$ and notice that $t^{k_1}_{k_2}(x^a_{2a}) S^{-1}(x^a_{1a}) = \delta^{k_1}_{k_2}$. Therefore

$$(\ell_{h_i} - \ell^{\mathcal{R}}_{h_i})(da x^a_{2a}) S^{-1}(x^a_{1a}) = \sum_{k_1=k_2} DI_i^k(a) = \sum_{k_1=k_2} d(M_i^j)(f_j^k * a) - (a * f_i^j) d(M_j^k). \quad (8.32)$$

Now $f_{j_1 j_2}^{k k}$ gives a non-trivial matrix $Y^{j_1}_{j_2}$, while $M_{j_1 j_2}^{k k}$ will become the Kronecker symbol $I \delta^{j_1}_{j_2}$ which vanishes as it is acted upon by d . This is what remains if the Defect Index were zero:

$$d(M_i^j)(Y_j * a) = 0$$

for all $a \in A$ — an incorrect statement!

9 Discussion

We have seen that the bicovariant differential calculus studied so far admits a natural contraction operator along generic vector fields which leads to a Lie derivative ℓ that is not unique. We stress that there are no problems when we deal with vector fields and functions only. This should suffice in many physics applications (e.g. gauge theories) that do not make direct reference to volume forms. To evaluate the defect index we have used the Leibniz rule for the exterior differential d [see (8.27) and (8.28)], this shows that a possible way to obtain a differential calculus with $\ell = \ell^{\mathcal{R}}$ is to define a new exterior differential d that satisfies a deformed Leibniz rule [17]. The problem of the non-unique Lie derivative adds to another mystery [18] of differential calculi on quantum groups: the number of forms is generally larger than one should expect from the classical limit. Notice also that the Defect Index is strictly connected with the impossibility of the Lie derivative to be a good differential operator i.e. satisfying a deformed Leibniz rule. Indeed, consider $V_k \equiv M_k^j t_j$; since V_k has a tensor index we would like ℓ_{V_k} to exhibit a nice (braided)

deformation of Leibniz rule. Using 5) of Theorem 7 we find

$$\begin{aligned}\ell_{V_i}(a\vartheta') &= (\ell_{V_i}a)\vartheta' + (a * f_i^j)\ell_{V_j}\vartheta' + [dM_i^j f_j^k * a - a * f_i^j dM_j^k] \wedge i_{t_k}(\vartheta') \\ &= (\ell_{V_i}a)\vartheta' + (a * f_i^j)\ell_{V_j}\vartheta' + DI_i^k(a) \wedge i_{t_k}(\vartheta').\end{aligned}\tag{9.33}$$

It is the Defect Index that is responsible for the appearance of the contraction operator in the Leibniz rule for the Lie derivative.

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